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2001 J. Phys. A: Math. Gen. 34 7527

(http://iopscience.iop.org/0305-4470/34/37/308)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 7527–7539 PII: S0305-4470(01)25348-7

Properties of the $\pm J$ Ising spin glass on the triangular lattice

J Poulter[1] **and J A Blackman**[2]

[1] Department of Mathematics, Faculty of Science, Mahidol University, Rama 6 Road, Bangkok 10400, Thailand
[2] Department of Physics, University of Reading, Whiteknights, PO Box 220, Reading RG6 6AF, UK

Received 29 May 2001
Published 7 September 2001
Online at stacks.iop.org/JPhysA/34/7527

Abstract
We study the ground state of the $\pm J$ Ising spin glass on the triangular lattice with an arbitrary concentration p of negative bonds. In agreement with a previous study on the square lattice, it is found that the spin correlation exponent η remains constant over a range of defect concentration $p_{c1} < p < p_{c2}$. Furthermore, the value of η obtained is 0.33 ± 0.02 which agrees, within error bars, with the square lattice result. This indicates that the spin glass phases on the two lattices are in the same universality class. However, the model on the triangular lattice has an upper critical concentration at $p = p_{c2}$ that is not present for the square lattice. This transition is found to be due to a percolation of frustrated plaquettes which drives a crossover to a frustrated antiferromagnetic phase. We show that this occurs at $p_{c2} = 0.8472 \pm 0.0001$ with the usual percolation exponents for two dimensions.

PACS numbers: 05.50.+q, 64.60.Cn, 75.10.Nr

1. Introduction

The short-range $\pm J$ Ising system is one of the most widely studied models of a spin glass. The exchange interactions J_{ij} are quenched random variables of fixed magnitude J but random sign. Specifically, the distribution is given by

$$P(J_{ij}) = p\delta(J_{ij} + J) + (1 - p)\delta(J_{ij} - J), \tag{1}$$

where $p \in [0, 1]$. The Hamiltonian is of the usual Edwards–Anderson [1] form, with only nearest-neighbour bonds. Our present understanding of this model and spin glasses in general, is contained in a number of reviews [2–5].

On a square lattice, there is a general consensus that for $p = 0.5$ a spin glass does exist at zero temperature. However, whether it exists for small finite temperatures is still a point of contention. One body of work [6–9] finds that the spin glass transition occurs at a temperature $T_c \sim 0.23J$, while others [10, 11] present contrary evidence. For $p \neq 0.5$, early indications [12] that the $T = 0$ spin glass is maintained have since been confirmed [13] on the square lattice. A spin correlation exponent $\eta = 0.34 \pm 0.02$, defining an algebraic decay in correlations

$$\left[\langle S_0 S_R \rangle^2\right]_{\mathrm{av}} \sim R^{-\eta} \tag{2}$$

was found to be unchanged for p down to 0.115.

The Ising model in two dimensions, without any magnetic field, is particularly susceptible to solution by combinatorial methods. One such method is the Pfaffian method [14, 15] which is perhaps best suited to solve models with disorder. Essentially, the model consists of non-interacting lattice fermions, and can also be formulated in terms of Grassmann variables [16, 17]. An account of the $\pm J$ Ising model in fermionic language has been given a while ago [18], and its potential for probing the zero-temperature limit fully explored. More recently the relevance of random bond Ising models, as ensembles of fermions, to disordered superconductors and quantum Hall states has been introduced [19, 20].

There have not been many studies of the $\pm J$ Ising model on the triangular lattice. Published results include numerical estimates of the ground-state energy [21] and some estimates of the defect concentration p_{c1} where ferromagnetism disappears. These are $p_{c1} \approx 0.10$–0.15 [22] and $p_{c1} \approx 0.165$–0.170 [23]. Nevertheless, knowledge of the fully frustrated case ($p = 1$) is extensive. In particular, the entropy has been calculated by Wannier [24] and the spin correlation exponent by Stephenson [25] who derived the asymptotic form

$$\langle S_0 S_R \rangle \sim R^{-1/2} \cos(2\pi R/3). \tag{3}$$

Clearly this suggests that there has to be a crossover between the spin glass and the fully frustrated model at some concentration between p_{c1} and 1. The idea of such a crossover has been suggested [26] but, to our knowledge, no evidence has been previously available. We are able to demonstrate the existence of a unique upper critical concentration, p_{c2}, that locates this crossover. At $p = 1$ all plaquettes are frustrated. As p is decreased the number of frustrated plaquettes is reduced until a percolation threshold is reached. The spin glass regime occurs at concentrations below this percolation threshold. We showed earlier that a characteristic feature of the spin glass regime is the presence of extended states within a fermion formalism.We find that as we increase p from within the spin glass regime, the extended states disappear at precisely the concentration that marks the percolation threshold, thus enabling us to define a unique value for p_{c2}. The crossover then is driven by a percolation of frustrated plaquettes, but it should be emphasized that this concept is quite distinct from that of frustrated percolation [27].

2. Formalism

The full details of the formalism as applied to the square lattice have been given earlier [18]. The adaptation to the triangular lattice is straightforward and only a brief outline is given here. The partition function is written as [14]

$$Z = 2^N \left[\prod_{\langle ij\rangle} \cosh(J/kT)\right] (\det D)^{1/2} \tag{4}$$

where the product is over all bonds on the N site lattice and the matrix D is real, skew-symmetric and of order $6N$. The Pfaffian is precisely the square root of the determinant of D. Using a real unitary transformation, D can be cast in 2×2 block diagonal form with

$$D \mid \alpha\rangle = -\epsilon \mid \beta\rangle \qquad D \mid \beta\rangle = \epsilon \mid \alpha\rangle. \tag{5}$$

We refer to the ϵ as eigenvalues which are $3N$ in total. However only a subset, which we call defect eigenvalues, contributes to the physics of the $\pm J$ model ground state. The number of defect eigenstates is exactly equal to the number of frustrated plaquettes in the system. In the $T \to 0$ limit these eigenstates are confined to those nodes of the lattice that are vertices of the frustrated plaquettes. Their eigenvalues take the form

$$\epsilon = (2/3)X \exp(-2rJ/kT) \tag{6}$$

where X is a real number and r is an integer. For a particular configuration, X and r can be determined exactly by using degenerate state perturbation theory [18], with r coinciding with the order at which the eigenvalue emerges. The changes in ground-state energy and entropy due to the introduction of frustration are respectively

$$\Delta F = 2J \sum_d r_d \tag{7}$$

$$\Delta S = k \sum_d \ln X_d \tag{8}$$

where the sums extend over all defect eigenvalues. For the fully frustrated model, the total energy is $-3JN + \Delta F = -NJ$ and the entropy is as reported by Wannier [24].

Concerning the eigenstates $|\alpha\rangle$ and $|\beta\rangle$, it can be easily shown that they are not unique since the respective eigenvalue is unchanged by an arbitrary rotation of the basis. Nevertheless it is possible to define [13] the spatial extent of an eigenstate through an invariant measure, l, according to

$$l = \langle x\rangle + \langle y\rangle \qquad\qquad \langle s\rangle = (P^2 + Q^2)^{1/2} \tag{9}$$

where s is x or y, and $P = \langle\alpha|s|\alpha\rangle - \langle\beta|s|\beta\rangle$ and $Q = 2\langle\alpha|s|\beta\rangle$. To understand these equations, note that the eigenstates are given in the basis of localized frustrated plaquette states [18], each of which has some coordinates (x, y). Note also that P is the Manhattan spatial extent. The overlap Q is often zero or otherwise small.

The importance of this spatial extent is seen from its distribution function which, if a spin glass exists, we expect to take the form [13, 18]

$$N(l) \sim L^2 l^{-\rho}. \tag{10}$$

Specifically, $N(l)$ is the number of eigenstate pairs with spatial extent larger than l for a lattice of linear dimension L. The exponent ρ provides a signature [18] of the nature of the states: if $\rho > 2$ all are localized, while $\rho < 2$ indicates the presence of extended states. Essentially, it is the appearance of extended states which destroys ferromagnetic order and induces a crossover to a spin glass [13]. It is also argued that ρ is related to the spin correlation exponent by $\eta = 2 - \rho$. Delocalization, of fermions, in the Ising model context, has also been discussed elsewhere [19, 20], although it is not yet clear how the arguments relate to the analysis given here.

3. Localized and extended states

We have collected data for a triangular lattice in the spin glass regime for three values of sample size L (64, 128 and 256). Typically, 512 samples were used in each configurational average. The boundary was of triangular shape to ensure that the entropy converged correctly to the result of Wannier [24]. Bonds on the boundary were set to infinite J which is equivalent to nesting the sample in an infinite unfrustrated lattice. This scheme has the advantage of preventing the system from using the boundary to counter the effects of frustration and also

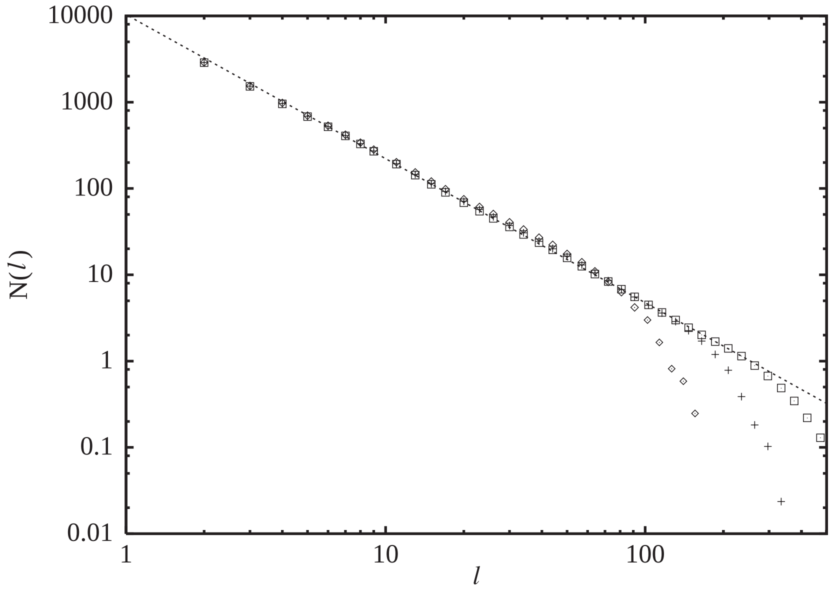

Figure 1. Distribution $N(l)$ (normalized to an $L = 256$ sample size) for $p = 0.5$ for samples with $L = 64$ (diamonds), 128 (pluses) and 256 (squares). A straight line fit to the $L = 256$ data is also shown.

avoids the complexities of using periodic conditions with fermions. Figure 1 shows the distribution function $N(l)$ for $p = 0.5$. A least squares fit of the data gives $\rho = 1.67 \pm 0.02$. This agrees, within error bars, with the value for the square lattice [13] where the data was less well conditioned. Figures 2 and 3 show the data for $p = 0.2$ and 0.175 with the fit for $p = 0.5$ also drawn for comparison. The value of the exponent ρ has remained constant. Figure 4 shows data for $L = 256$ and four values of concentration p (0.17, 0.16, 0.15 and 0.10) compared with the $p = 0.175$ fit. A fit at $p = 0.17$ is possible but with larger gradient and error bars. This suggests that the spin glass exists for $p > p_{c1}$, where p_{c1} lies between 0.170 and 0.175. Further evidence is forthcoming from the data for the entropy which can be fitted to the expression

$$S(L) = S + \beta L^{-1}. \tag{11}$$

We find that the coefficient β is very small for $p = 0.175$, positive for $p > 0.175$ and negative otherwise. Basically, the effect of the boundary tends to increase the degeneracy of the ground state in the presence of extended states. A similar result was observed on the square lattice [13]. We conclude that p_{c1} is in the range 0.170–0.175 which overlaps with the result of Achilles *et al* [23]

Looking above $p = 0.5$, figures 5 and 6 show the distribution function for $p = 0.75$ and 0.8. The gradients of the fits are the same as for $p = 0.5$ although we have to focus on the asymptotic regime ($l > \sim 30$) for $p = 0.8$. The data for $p = 0.83$ is shown in figure 7. Power law behaviour can still be seen for ($l > \sim 80$) in the $L = 256$ data with the gradient of the straight line fit unchanged, but we are close to the limits of the useful data due to the finite size of the sample. Finally, the data for $p = 0.84$ is shown in figure 8, where the gradient of the fit is 1.79 ± 0.08, but again we have a limited range due to the finite sample size. Clearly, we are close to an upper critical concentration p_{c2} at around 0.84. In the concentration range,

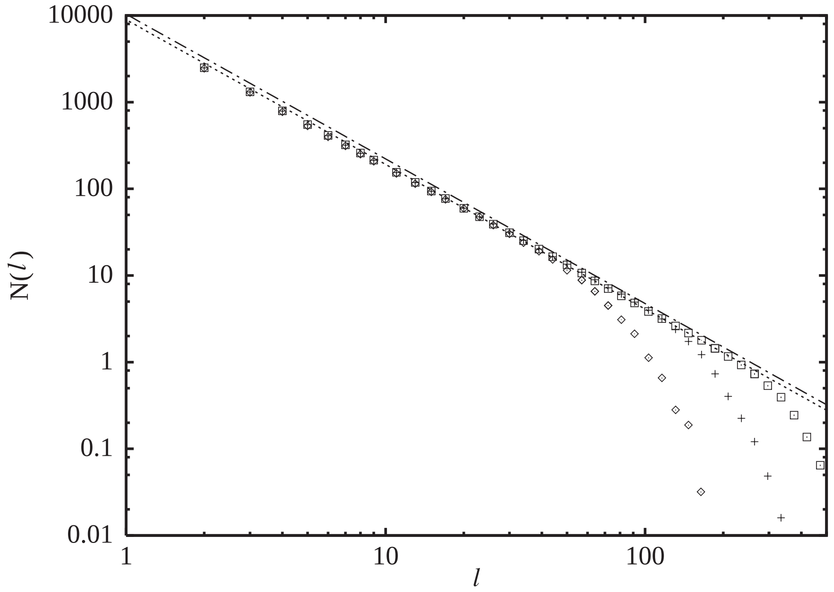

Figure 2. As figure 1 for $p = 0.2$. The dotted line is the straight line fit to the $L = 256$ data; the fit for $p = 0.5$ (dot–dash line is included for comparison).

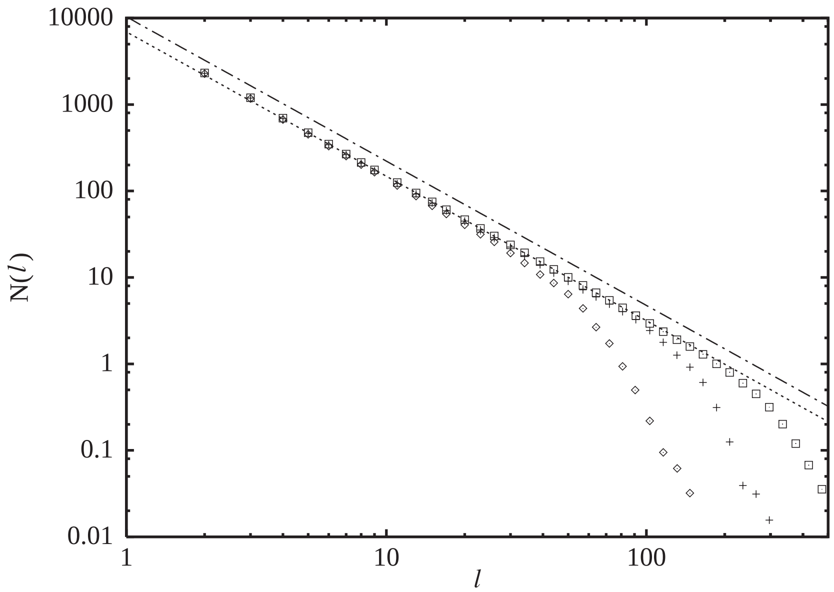

Figure 3. As figure 2 for $p = 0.175$.

$p_{c1} < p < p_{c2}$, a spin glass phase exists characterized by a universal exponent ρ. A different type of phase occurs for $p > p_{c2}$, and examining that regime will give us a more precise estimate of p_{c2}.

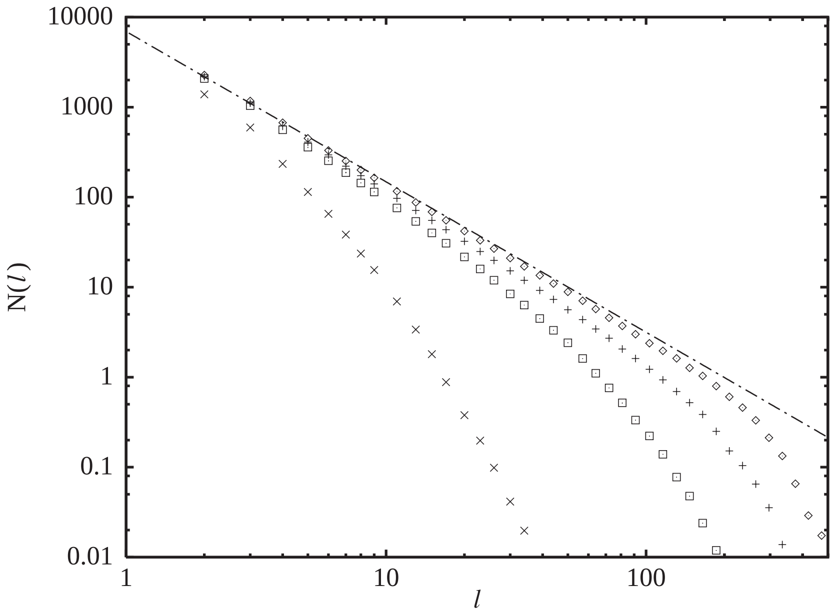

Figure 4. Distribution $N(l)$ for $L = 256$ samples for $p = 0.17$ (diamonds), 0.16 (pluses), 0.15 (squares) and 0.10 (crosses). The line fit for $p = 0.175$ is also shown.

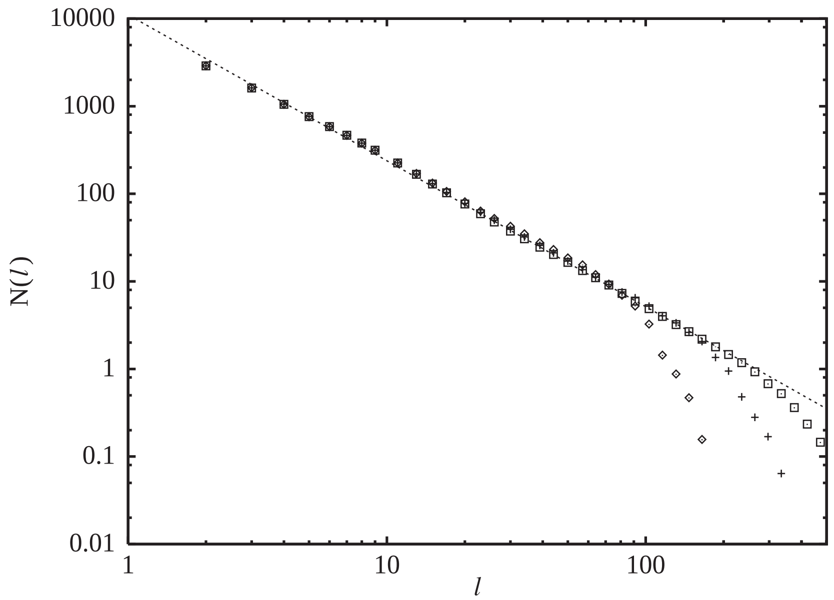

Figure 5. As figure 1 for $p = 0.75$.

4. Percolation

We now consider the high-concentration regime. At $p = 1$, all plaquettes on the lattice are frustrated. For lower values of p, we study clusters of frustrated plaquettes. A cluster

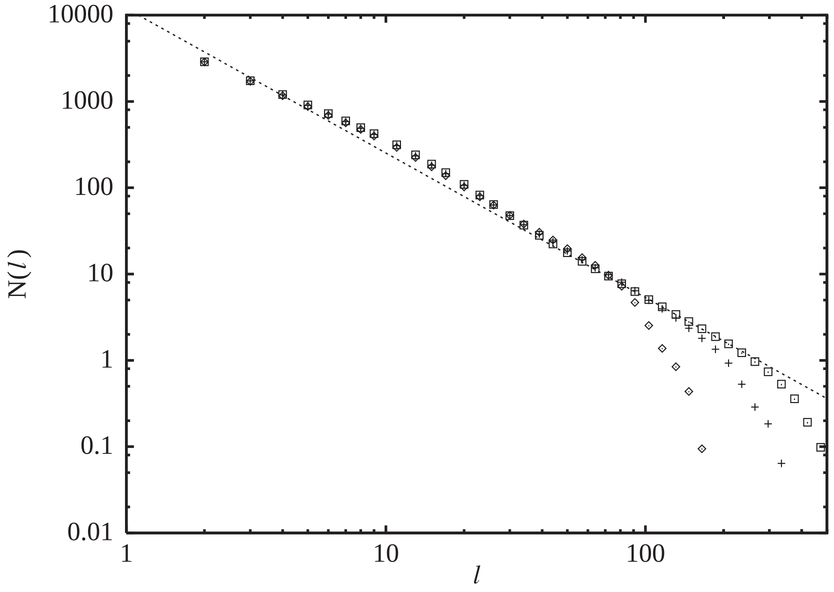

Figure 6. As figure 1 for $p = 0.8$.

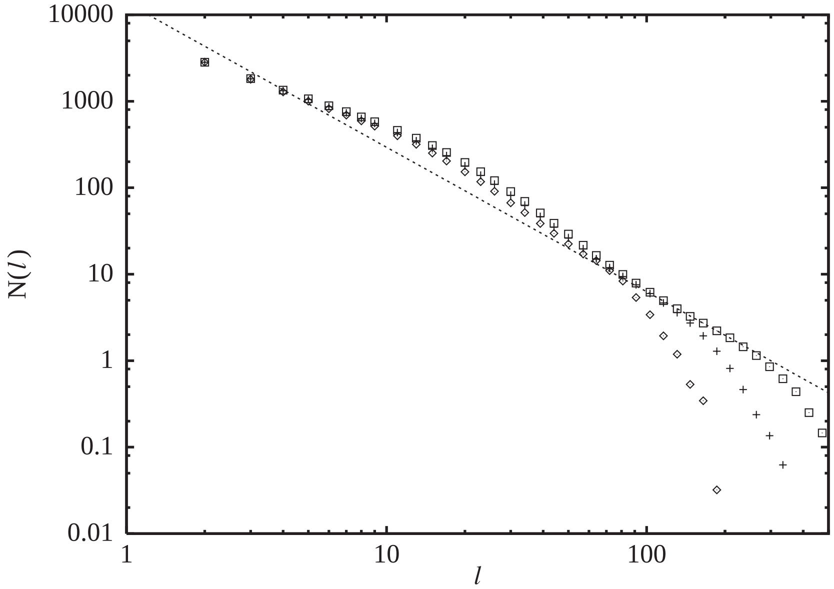

Figure 7. As figure 1 for $p = 0.83$.

is defined as a set of frustrated plaquettes that are nearest-neighbours to at least one other member of the cluster. For p below the percolation threshold, no spanning clusters exist. The size of a cluster is measured by the number of frustrated plaquettes it contains. Percolation occurs if the largest cluster is comparable to the sample size.

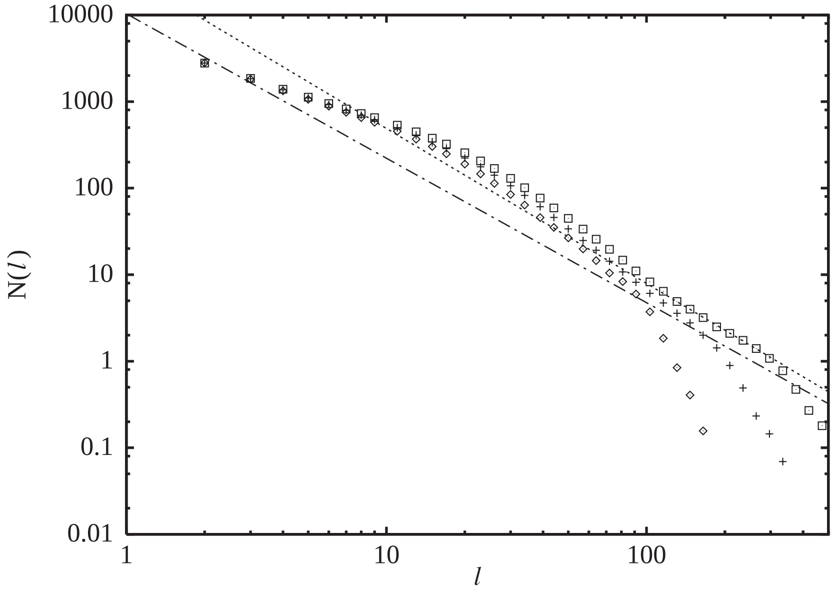

Figure 8. As figure 2 for $p = 0.84$.

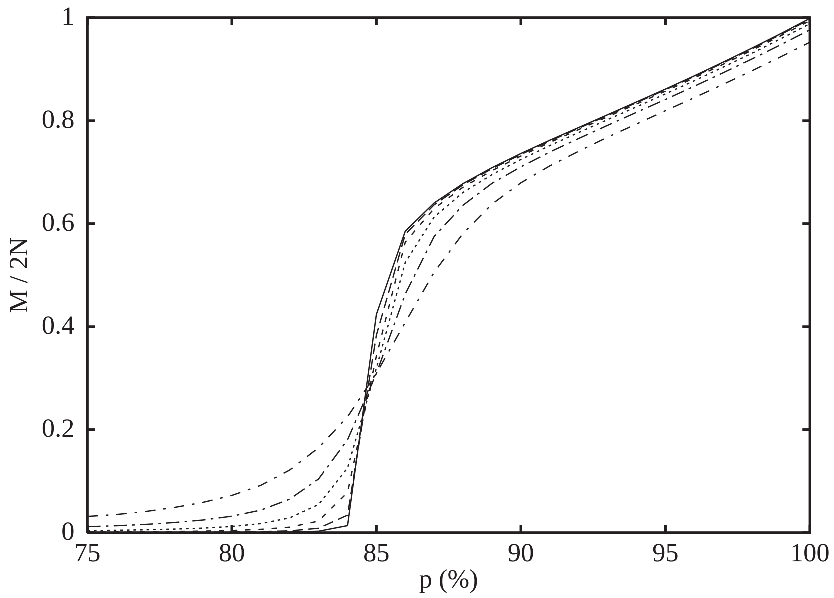

Figure 9. Largest cluster size (normalized to total number of plaquettes, $2N$), as a function of p for sample sizes $L = 64$, 128, 256, 512, 1024 and 2048. The transition is sharper for larger L.

Figure 9 shows the largest cluster size M for various sample dimensions L (64, 128, 256, 512, 1024 and 2048). M is the average over the largest clusters of a large number of configurations. In the figure, M is expressed as a fraction of the total number of plaquettes in the sample, $2N$. There is a rapid increase where we expect the crossover to occur. The

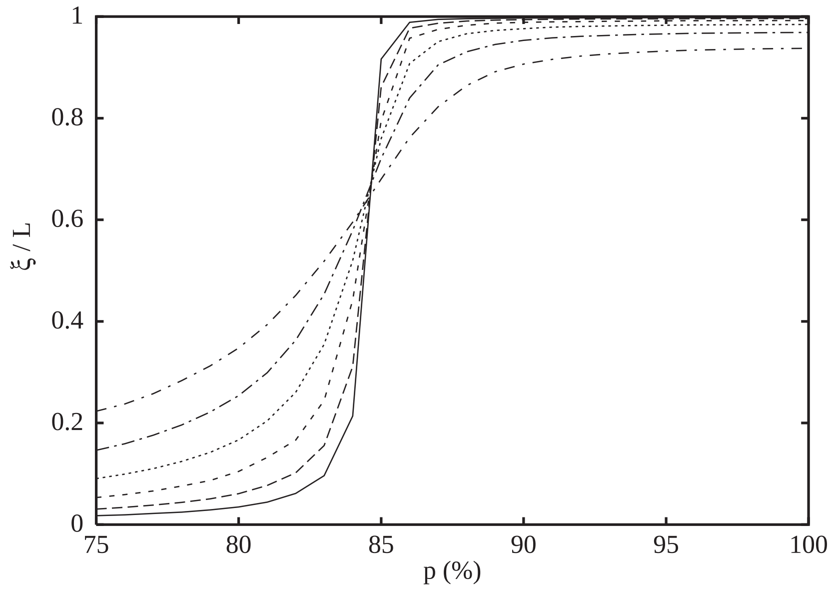

Figure 10. As figure 9 for the correlation length (normalized to linear size of the sample).

correlation length is also studied. This can be defined as $\xi = (L_x + L_y)/2$, where the largest cluster has linear dimensions L_x and L_y. This is shown in figure 10, where evidence of the percolation threshold is again obvious.

In order to make contact with the scaling behaviour of percolation theory and to obtain a very accurate estimate for p_{c2}, we use finite-size scaling for the correlation length, ξ, the largest cluster size, M, and also the mean cluster size, S, defined in the usual way

$$S = \frac{\sum_s s^2 n_s}{\sum_s s n_s}. \tag{12}$$

The sums are over cluster size and n_s is the number of clusters of size s. By analogy with standard percolation theory [28], we use the following finite-size scaling equations:

$$\xi \sim L f_\xi\big[(p - p_{c2})L^{1/\nu}\big] \tag{13}$$

$$M \sim L^{-\beta/\nu} f_M\big[(p - p_{c2})L^{1/\nu}\big] \tag{14}$$

$$S \sim L^{\gamma/\nu} f_S\big[(p - p_{c2})L^{1/\nu}\big] \tag{15}$$

where f_ξ, f_M and f_S are universal functions. We ignore the issue of excluding the largest cluster, a common procedure in finite scaling analysis in percolation theory. It is not straightforward to do, and excellent data collapse occurs anyway.

The percolation of frustrated plaquettes on the triangular lattice is actually a site percolation problem on a hexagonal lattice, but with special rules. Nevertheless, we expect and find that the percolation exponents are the same, that is $\beta = 5/36$, $\gamma = 43/18$ and $\nu = 4/3$. Figures 11–13 show the data collapses for ξ, M and S with data taken for several sample sizes L (128, 256, 512, 1024 and 2048). These data collapses are very sensitive to changes in p_{c2} or the percolation exponents. If we use the above values of β, γ and ν, the optimum data collapse gives the threshold with a remarkable level of accuracy. The value of p_{c2} obtained is 0.8472 ± 0.0001.

This estimate of p_{c2} is entirely consistent with the value of the critical concentration determined in the previous section. There we found that ρ retained a constant value of

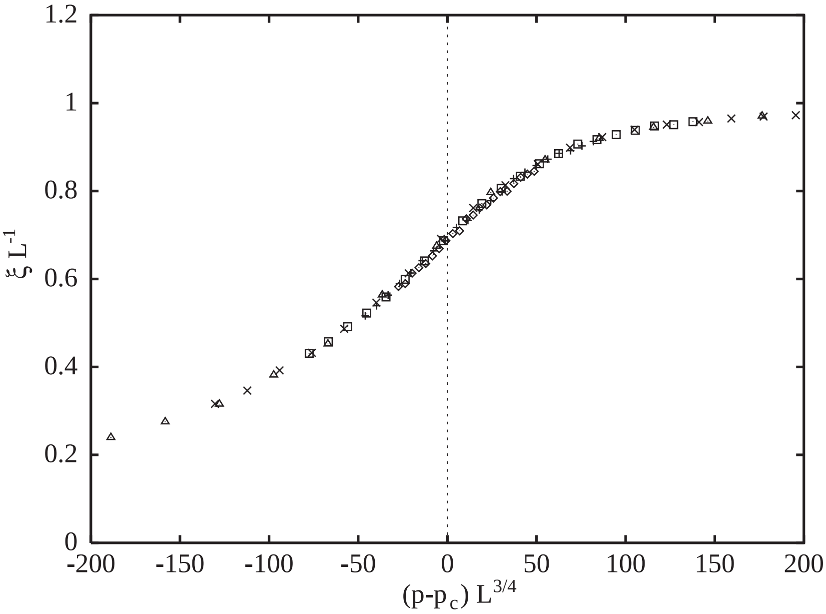

Figure 11. Data collapse for the correlation length, using equation (13). Sample sizes are $L = 128$ (diamonds), 256 (pluses), 512 (squares), 1024 (crosses), and 2048 (triangles).

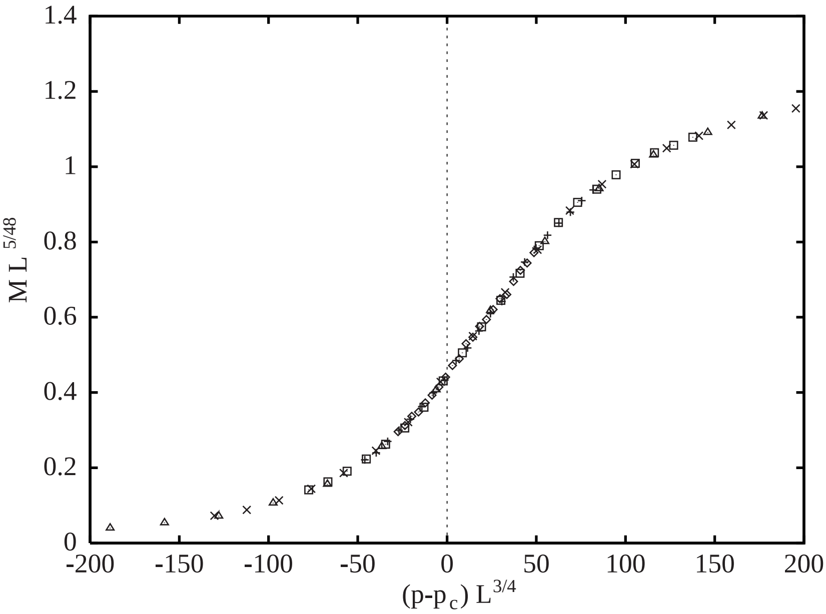

Figure 12. Data collapse for the largest cluster size, using equation (14). Symbols are as for figure 11.

1.67 ± 0.02 up to a concentration of about 0.83. By $p = 0.84$, ρ had increased to 1.79 ± 0.08, a very strong indication that it is tending towards a value of 2 at p_{c2}. The critical value, $\rho = 2$, marks the disappearance of extended states.

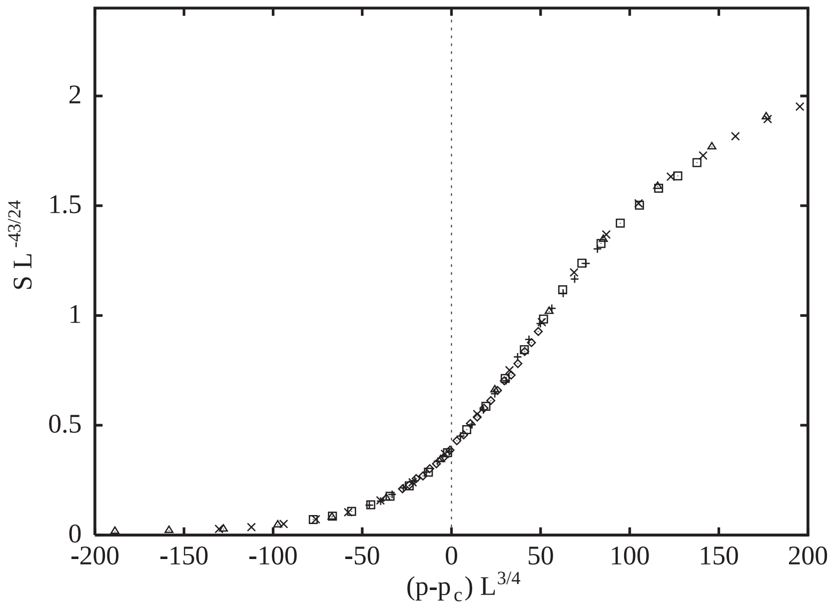

Figure 13. Data collapse for the mean cluster size, using equation (15). Symbols are as for figure 11.

5. Energy and entropy

Results for the ground-state energy and entropy as calculated from equations (7) and (8) are given in table 1. The values were obtained at different lattice sizes and extrapolated to infinity. The energy values compare well with other work [21]. We have not been able to obtain reliable data for $p > p_{c2}$ since the crossover is also characterized by a considerable increase in memory requirement. Nevertheless rough estimates have been obtained [29] for the entropy by only computing the contribution from first-order degenerate state perturbation theory [18] with sparse matrix techniques. However, the entropy is a smooth function of p even in the vicinity of the crossover. Closer to $p = 1$ the data are consistent with the analytical result [30] that the entropy differs from the result of Wannier by $-3(1-p)\ln 3$.

6. Conclusions

The key results of this work are the identification of two critical concentrations, p_{c1} and p_{c2}, and the characterization of two phases. Finite size scaling yielded an extremely accurate estimate for p_{c2} of 0.8472 ± 0.0001, the threshold for percolation of frustrated plaquettes. At $p = 1$, the fully frustrated antiferromagnet is known to be in a universality class with other fully frustrated systems such as the Villain model [31–33].We propose that the essential feature of this universality class is an infinite cluster of frustrated plaquettes. In which case, all triangular systems with $p_{c2} < p < 1$ will also be members, with a correlation function exponent of $1/2$ as in equation (3), although the oscillatory factor in that equation is obviously special to the fully frustrated system. It is now fairly well established [34] that the critical behaviour of the Ising ferromagnet on dilution with ferromagnetic bonds of different magnitude is essentially unchanged from that of the pure system. Randomness alone does no more than

Table 1. Ground-state energy (in units of J) and entropy (in units of k) per spin for selected values of p. The uncertainty in the last decimal place is indicated by the figures in parentheses.

p	Energy	Entropy
0.100	−2.4070 (3)	0.01077 (1)
0.110	−2.3497 (2)	0.01389 (2)
0.120	−2.2940 (4)	0.01759 (4)
0.130	−2.2403 (4)	0.02156 (3)
0.140	−2.18767 (1)	0.025861 (2)
0.150	−2.13855 (2)	0.03027 (3)
0.160	−2.0918 (2)	0.03424 (2)
0.170	−2.0495 (3)	0.03746 (3)
0.175	−2.03052 (8)	0.03893 (4)
0.180	−2.01264 (8)	0.04018 (4)
0.190	−1.97740 (9)	0.04293 (7)
0.200	−1.9471 (4)	0.04532 (5)
0.250	−1.83237 (6)	0.05473 (5)
0.300	−1.76593 (8)	0.06004 (5)
0.350	−1.7299 (2)	0.0636 (1)
0.400	−1.71389 (8)	0.0647 (1)
0.450	−1.7091 (2)	0.065028 (5)
0.500	−1.7085 (1)	0.065035 (2)
0.550	−1.70746 (6)	0.06514 (1)
0.600	−1.7030 (1)	0.06552 (4)
0.650	−1.6927 (2)	0.0664 (2)
0.700	−1.6732 (2)	0.06782 (3)
0.750	−1.6406 (1)	0.0707 (2)
0.800	−1.5926 (2)	0.0754 (2)
0.810	−1.58136 (1)	0.076438 (8)
0.820	−1.56797 (7)	0.07818 (9)
0.830	−1.55605 (9)	0.0793 (1)

introduce small finite size corrections. We are not aware of similar work on systems involving frustration, but would expect analogous behaviour to occur.

In the concentration range $p_{c1} < p < p_{c2}$, the exponent ρ that signals the presence of extended states remains constant at 1.67 ± 0.02, implying a value for η of 0.33 ± 0.02. This is consistent with values found earlier for the square lattice [13], and establishes a universality class for these systems. There does remain a question about the crossover region itself. The value of p_{c2} as approached from above is defined extremely precisely. However, the constancy of ρ was seen up to about $p = 0.83$, about 0.01 below p_{c2} and then, as p approaches p_{c2}, it increased towards the critical value of 2 that signals the disappearance of extended states. Similar behaviour also occurs in a very small region close to p_{c1}. Now, certainly the whole of the range $p_{c1} < p < p_{c2}$ can be characterized by the presence of extended states, but there is a possibility that the concentration range over which ρ and η are constant may be less than the given range by a very small amount. It seems more likely, however, that the deviation from constancy that we are observing very close to p_{c1} and p_{c2} is a finite size effect and, for an infinite system, ρ and η are constant throughout the range.

The approach we have taken in this work is specifically geared to zero temperatures and, as such, complements the bulk of work in this field which is done for finite temperatures, but which encounters major problems in extrapolating to the zero-temperature limit. Our method, by contrast, is exact at the $T = 0$ limit.

Acknowledgments

We thank the High Performance Computing Centre, NECTEC, Thailand for the use of computational facilities. One of us (JP) also acknowledges support from the Thailand Research Fund, Contract RTA/02/2542.

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